The transport equation and zero quadratic variation processes

Jorge Andrés Clarke De la Cerda

Departamento de Matemática
Universidad Técnica Federico Santa María, Valparaíso, Chile.
clarkemove@gmail.com

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joint work with professors Christian Olivera and Ciprian Tudor

Outline

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Harold Edwin Hurst

"Given a series of annual discharges recorded for a past period, it is easy to calculate what storage would have been enough to equalize the flow and send down the mean discharge every year throughout the period. Much more than this is needed, however, before a policy can be laid down for the future, since the past is never exactly repeated"

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THE PROBLEM OF LONG-TERM STORAGE IN RESE

H. E. Hurst[a]

[a] Ministry of Public Works, Egypt

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THE PROBLEM OF LONG-TERM STORAGE IN RESERVOIRS

H. E. Hurst, C.M.G., M.A., D.Sc., F.Inst.P.,
Scientific Consultant to the Ministry of Public Works, Egypt

1. INTRODUCTION

The use of the water of a river for irrigation so as to extract the maximum benefit requires that the flow of the river shall be regulated by means of reservoirs. The ideal would be that the river should be completely controlled so as to send down a constant annual discharge distributed throughout the year according to the seasonal requirements of the crops.

In the following discussion, which is based on two earlier papers by the author,[1, 2] seasonal variation of flow and crop requirements within the year are not considered, since only annual totals are used. The effects of these must be considered separately and added to those dealt with here.

Given a series of annual discharges recorded for a past period, it is easy to calculate what storage would have been enough to equalize the flow and send down the mean discharge every year throughout the period. Much more than this is needed, however, before a policy can be laid down for the future, since the past is never exactly repeated.

Long range dependence (LRD)

Long-range dependency is a phenomenon that may arise in the analysis of spatial or time series data. It relates to the rate of decay of statistical dependence, with the implication that this decays more slowly than an exponential decay, typically a power-like decay. LRD is often related to self-similar processes or fields.

Famous random fields in Geostatistics

The Gneiting family of space-time covariance functions.

- Gaussian
- Stationary

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$$K(h,u) = \sigma^2 \psi(u^2)^{-d/2} \phi\left(\frac{\|h\|^2}{\psi(|u|^2)}\right)$$

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Other families, Dagum, Cauchy, ...

Stochastic Partial Differential Equations

Let L be a partial differential operator, *e.g.* $L = \frac{\partial^2}{\partial t^2} - \Delta_{\mathbf{x}}$, $t \in]0, T]$, $\mathbf{x} \in \mathbb{R}^k$, and $u(t, \mathbf{x}) = (u_1(t, \mathbf{x}), \ldots, u_d(t, \mathbf{x})) \in \mathbb{R}^d$ the solution of the system

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\begin{cases}
Lu_1(t,\mathbf{x}) & = b_1(u(t,\mathbf{x})) + \sum_{j=1}^d \sigma_{1,j}(u_1(t,\mathbf{x}))\dot{W}^j(t,\mathbf{x}) \\
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- F. Lindgren, H. Rue & J. Lindstrom (2011): *An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach.*

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The linear transport equation,

$$\partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) = 0\,. \tag{1}$$

Emerges as a model for the concentration (density) of a pollutant in a flow. It is a particular case of the advection equation when the flow under consideration is incompressible (i.e. has zero divergence).

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The stochastic linear transport equation

We analyse the following one-dimensional Cauchy problem: given an initial-data u_0, find $u(t,x;\omega) \in \mathbf{R}$, satisfying

$$\begin{cases} \partial_t u(t,x;\omega) + \partial_x u(t,x;\omega) \left(b(t,x) + \dfrac{dZ_t}{dt}(\omega) \right) = 0, \\ u(t_0,x) = u_0(x), \end{cases} \tag{2}$$

with $(t,x) \in U_T = [0,T] \times \mathbf{R}$, $\omega \in \Omega$, $b : [0,T] \times \mathbf{R} \to \mathbf{R}$ a given vector field, and the noise $(Z_t)_{t \geq 0}$ is a stochastic process with zero quadratic variation.

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- E. Fedrizzi & F. Flandoli. (2013): *Noise prevents singularities in linear transport equations.*
- F. Flandoli, M. Gubinelli & E. Priola, (2010): *Well-posedness of the transport equation by stochastic perturbation.*
- H. Kunita (1984): *Stochastic differential equations and stochastic flows of diffeomorphisms.*

Outline

Stochastic Calculus via regularization

- T is a fixed positive real number.
- $(X_t)_{t\geq 0}$ a continuous process.
- $(Y_t)_{t\geq 0}$ a process with paths in $L^1_{loc}(\mathbb{R}^+)$, *i.e.* for any $b>0$, $\int_0^b |Y_t|dt < \infty$ a.s.

$$I^0(\epsilon, Y, dX)(t) = \int_0^t Y_s \frac{(X_{s+\epsilon} - X_{s-\epsilon})}{2\epsilon} ds, \quad t \geq 0. \tag{3}$$

Then, the simmetric integral is defined as

$$\int_0^t Y_s d^\circ X_s := \lim_{\epsilon\to 0} I^0(\varepsilon, Y, dX)(t), \tag{4}$$

for every $t \in [0,T]$, provided the limit exists in the *ucp* sense (uniformly on compacts in probability).

Stochastic Calculus via regularization

The covariation or generalized bracket, $[X, Y]_t$ of two stochastic processes X and Y is defined as the limit *ucp* when ε goes to zero of

$$[X, Y]_{\varepsilon,t} = \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s)\, ds, \quad t \geq 0.$$

Note that $[X, Y]$ coincide with the classical bracket when X and Y are semimartingales.

F. Russo & P. Vallois (1993): Forward, backward and symmetric stochastic integration.
F. Russo & P. Vallois (2007): Elements of stochastic calculus via regularization.

Malliavin Derivative

- $\mathcal{H}$ a real separable Hilbert space.
- $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space.
- $(B(\varphi), \varphi \in \mathcal{H})$ a centred Gaussian family of random variables such that

$$\mathbb{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}}$$

- $\mathcal{S}$ the space of smooth functionals of the form $F = g(B(\varphi_1), \ldots, B(\varphi_n))$.
  (g is a smooth function with compact support and $\varphi_i \in \mathcal{H}, i = 1, ..., n$)

For functions $F \in \mathcal{S}$ the malliavin derivative operator D, acts in the form

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i.$$

Malliavin Derivative

The operator D is closable from $\mathcal{S}$ into $L^2(\Omega;\mathcal{H})$, and accept the following chain rule:

Proposition

If $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function and $F \in \mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F) = \varphi'(F) DF. \tag{5}$$

where $\mathbb{D}^{1,p}$ which is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p}^p = \mathbb{E}F^p + \mathbb{E}\|DF\|_{\mathcal{H}}^p.$$

D. Nualart (2006): Malliavin Calculus and Related Topics.

Outline

Weak solution

Definition

A stochastic process $u \in L^\infty(\Omega \times [0,T] \times \mathbb{R})$ is called a weak L^p-solution of the Cauchy problem (2), if for any $\varphi \in C_c^\infty(\mathbb{R})$, $\int_\mathbb{R} u(t,x)\varphi(x)dx$ is an adapted real valued process which has a continuous modification, finite covariation, and for all $t \in [0,T]$, $\mathbb{P}$-almost surely

$$\begin{aligned}\int_\mathbb{R} u(t,x)\varphi(x)dx &= \int_\mathbb{R} u_0(x)\varphi(x)\,dx + \int_0^t\int_\mathbb{R} u(s,x)\,b(s,x)\partial_x\varphi(x)\,dxds \\ &+ \int_0^t\int_\mathbb{R} u(s,x)\,b'(s,x)\,\varphi(x)\,dxds + \int_0^t\int_\mathbb{R} u(s,x)\,\partial_x\varphi(x)\,dxd^\circ Z_s.\end{aligned}$$

where $b'(s,x)$ denotes the derivative of $b(s,x)$ with respect to the spatial variable x, and the integral $d^\circ Z$ is a symmetric integral defined via regularization (see (4)).

Weak solution

Proposition

Assume that $b \in L^\infty((0,T); C_b^1(\mathbb{R}))$. Then there exists a $C^1(\mathbb{R})$ stochastic flow of diffeomorhism $(X_{s,t},\ 0 \le s \le t \le T)$, that satisfies

$$X_{s,t}(x) = x + \int_s^t b(u, X_{s,u}(x))du + Z_t - Z_s \tag{6}$$

for every $x \in \mathbb{R}^d$. Moreover, given $u_0 \in L^\infty(\mathbb{R})$, the stochastic process

$$u(t,x) := u_0(X_t^{-1}(x)), \qquad t \in [0,T], x \in \mathbb{R} \tag{7}$$

is the unique weak $L^\infty-$solution of the Cauchy problem 2, where $X_t := X_{0,t}$ for every $t \in [0,T]$.

Weak solution

Lemma

Assume $b \in L^\infty\left((0,T); C_b^1(\mathbb{R})\right) \cap C\left((0,T)\times\mathbb{R}\right)$. *Then the inverse flow satisfies the backward stochastic equation*

$$Y_{s,t}(x) = x - \int_s^t b(r, Y_{r,t}(x))dr - (Z_t - Z_s) \tag{8}$$

for every $0 \le s \le t \le T$ *and for every* $x \in \mathbb{R}$.
Moreover, Y *is the unique process that satisfies (8) with* $Y_{s,s}(x) = x$.

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Remark

If we set $R_{t,x}(u) = Y_{t-u,t}(x)$, $B_t(a,x) = -b(t-a,x)$ and $Z_{u,t} = -(Z_t - Z_{t-u})$ for $t \in [0,T]$, $u \in [0,t]$, and $x \in \mathbb{R}$, then we have

$$R_{t,x}(u) = x + \int_0^u B_t(a, R_{t,x}(a))da + Z_{u,t}. \tag{9}$$

Outline

Malliavin differentiability of the inverse flow

$$b \in L^{\infty}\left((0,T); C_b^1(\mathbb{R})\right) \cap C\left((0,T) \times \mathbb{R}\right). \tag{10}$$

The noise Z is a zero quadratic variation process, adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ such that $Z_t \in \mathbb{D}^{1,2}$ for every $t \in [0,T]$.

$$\sup_{t\in[0,T]} \mathbb{E}\,|Z_t|^2 < \infty \text{ and } \sup_{t\in[0,T]} \mathbb{E}\|DZ_t\|^2_{L^2([0,T])} < \infty. \tag{11}$$

Lemma

Under hypothesis (10) and (11), the equation (9) has an unique solution.

Malliavin differentiability of the inverse flow

Proposition

Assume (10) and (11). Then, for $x \in \mathbb{R}, t \in [0,T], u \leq t$, the random variable $R_{t,x}(u)$ given by (9) belongs to $\mathbb{D}^{1,2}$. Moreover, the Malliavin derivative of $R_{t,x}(u)$ satisfies

$$D_\alpha R_{t,x}(u) = \int_0^u B_t'(s, R_{t,x}(s)) D_\alpha R_{t,x}(s) ds + D_\alpha Z_{u,t} \tag{12}$$

for every $\alpha < t$.

Density of the Solution

Proposition

Assume that b satisfies (10), Z satisfies (11) and Y be given by (8). Then for every $s \leq t, \alpha \leq t$ and $x \in \mathbb{R}$, we have

$$\begin{aligned} D_\alpha Y_{s,t}(x) &= 1_{(0,T)}(\alpha) e^{-\int_s^t b'(u, Y_{u,t}(x)) du} \\ &\quad \times \int_s^t b'(u, Y_{u,t}(x)) D_\alpha(Z_{u,t}) e^{\int_u^t b'(r, Y_{r,t}(x)) dr} \, du \\ &\quad + D_\alpha(Z_{s,t}). \end{aligned} \tag{13}$$

Density of the Solution

Proposition

Fix $0 \leq s \leq t \leq T$. Assume (10) and (11). In addition we will suppose that for every $0 < s < t \leq T$

$$\|DZ_t\|^2_{L^2([t-s,T])} = \int_{t-s}^{T} (D_\alpha Z_t)^2 d\alpha > 0, \text{ almost surely.} \tag{14}$$

Then the law of $Y_{s,t}(x)$ is absolutely continuous with respect to the Lebesgue measure.

Density of the Solution

Theorem

Let $u(t,x)$ be the solution to the transport equation (2). Assume that $u_0 \in C^1(\mathbb{R})$ such that there exists $C > 0$ with $(u_0'(x))^2 > C$ for every $x \in \mathbb{R}$. Then, for every $t \in [0,T]$ and for every $x \in \mathbb{R}$, the random variable $u(t,x)$ is Malliavin differentiable. Moreover $u(t,x)$ admits a density with respect to the Lebesgue measure.

Outline

The Hermite Process

We denote by $(Z_t^{(q,H)})_{t\in[0,T]}$ the Hermite process with *self-similarity parameter* $H \in (1/2, 1)$. For $t \in [0, T]$ it is given by

$$Z_t^{(q,H)} = d(H) \int_0^t \dots \int_0^t \mathrm{d}W_{y_1} \dots \mathrm{d}W_{y_q} \left(\int_{y_1 \vee \dots \vee y_q}^t \partial_1 K^{H'}(u, y_1) \dots \partial_1 K^{H'}(u, y_q) \mathrm{d}u \right)$$

the kernel $K^H(t, s)$ has the expression

$$K^H(t, s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} \, \mathrm{d}u$$

where $t > s$ and $c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-1/2)} \right)^{1/2}$ and $\beta(\cdot, \cdot)$ is the Beta function. For $t > s$, the kernel's derivative is

$$\frac{\partial K^H}{\partial t}(t, s) = c_H \left(\frac{s}{t} \right)^{1/2-H} (t-s)^{H-3/2}$$

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The Hermite Process

Lemma

Z^H is a zero quadratic variation process.

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The Hermite process satisfies (11) and (14).

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Thanks